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THE THEORY OF MIXTURE CURVES

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A "Mixture" curve is generated on probability paper(such as Weibull paper) whenever the individual data points represent random selections from a collection (total population) consisting of two or more distinct sub-populations.

The "Mixture" curve represents the total POPULATION picture generated by mixing the SUB - POPULATIONS in specific PROPORTIONS .

For example, look at FIGURE 1 below.

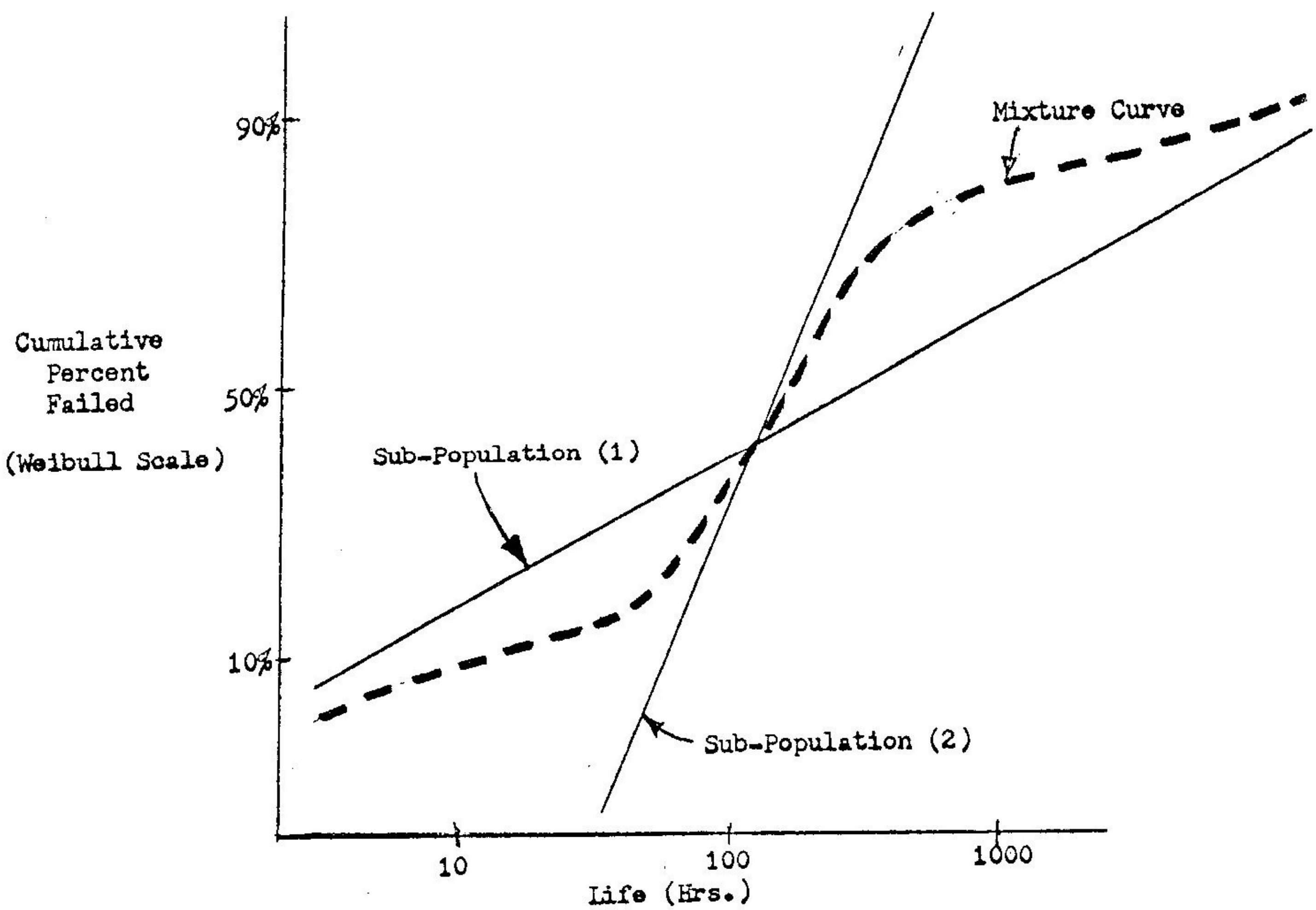

FIGURE 1

In this illustration, sub-population (1) and sub-population (2) are mixed together in the data collection, and even though parts (1) and (2) have straight Weibull plots, the "Mixture" curve is not straight. The S-shaped dotted curve is the resulting graphical picture of the entire data collection.

A "Mixture" curve generated from straight Weibull lines can never be STRAIGHT .

Another shape of "Mixture" curve is generated when sub-populations (1) and (2) have the same Weibull slope, as in FIGURE 2 below.

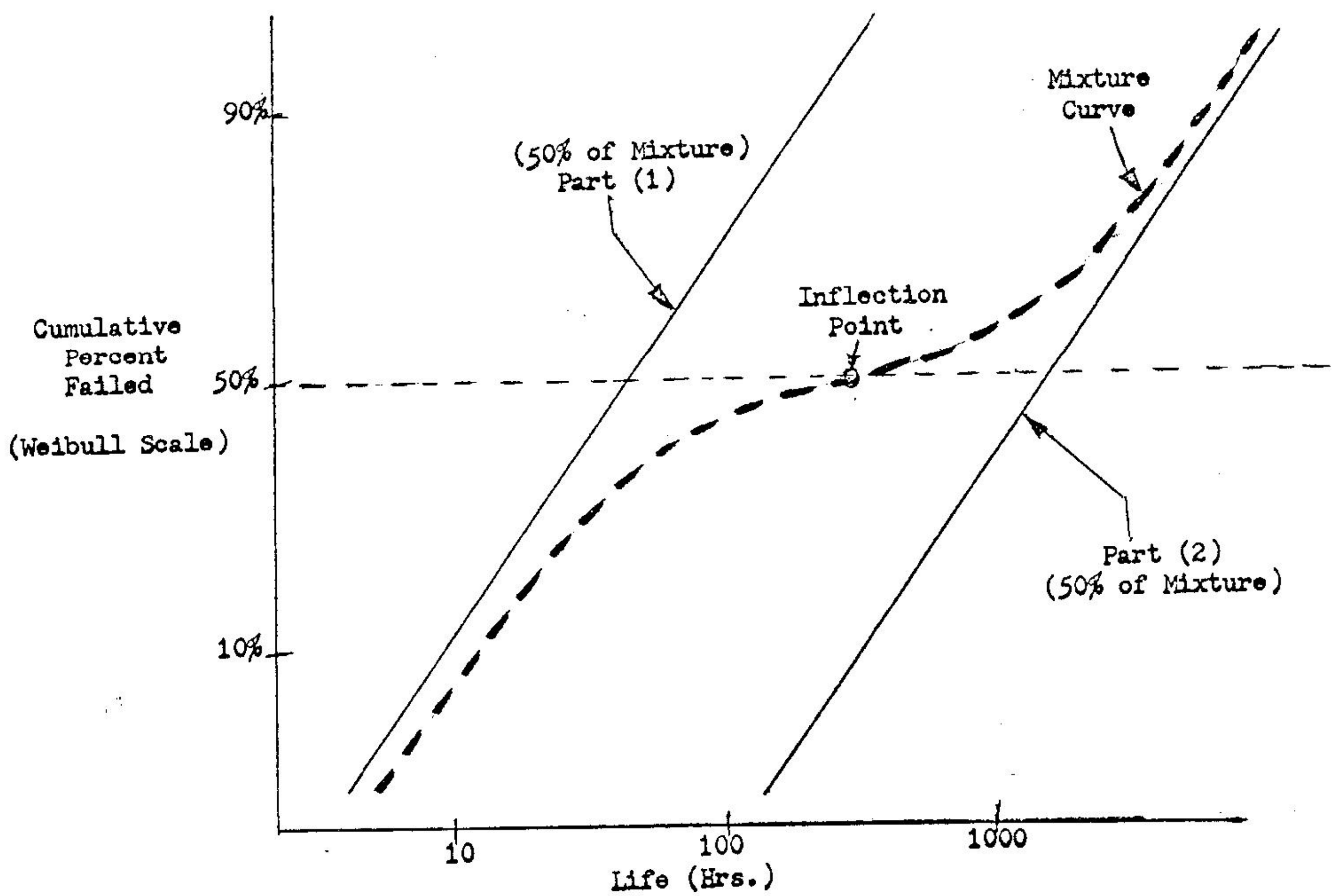

FIGURE 2

Curves of the Type in FIGURE 2 :

$$\left\{\begin{array}{l} F_1(x) = 1 - EXP(-(x/\theta_1)^b) \\ F_2(x) = 1 - EXP(-(x/\theta_2)^b) \end{array}\right\}$$ (Note the same Weibull slope) (b in both.)

$$F(x) = 1 - Q_1\, EXP(-(x/\theta_1)^b) - Q_2\, EXP(-(x/\theta_2)^b)$$

$$(Q_1 + Q_2 = 1)$$

In curves of the type in FIGURE 2, the mixture curve reaches quantile Q_1 at

$$x = \theta_2 \left(\ln \frac{1}{\xi}\right)^{1/b} = \text{Abscissa of Inflection Point,}$$

where ξ satisfies the equation

$$\left(\frac{Q_1}{1 - Q_1}\right) \xi^{\left(\frac{\theta_2}{\theta_1}\right)^b} + \xi = 1 .$$

NOTE : THE ORDINATE OF THE INFLECTION POINT IS ALWAYS Q_1 .

In FIGURE 2 , the inflection point of the "Mixture" curve is at the 50% level. This is because Part (1) makes up 50% of the total collection.

In general, if Part (1) (on the left) makes up the fraction Q_1 of the total collection, then the inflection point of the "Mixture" curve will be at quantile level Q_1 .

A point of inflection is a transition point from negative CURVATURE to positive CURVATURE .

In FIGURE 1 the "MIXTURE" CURVE has small slope at its ENDS, while in FIGURE 2 the "Mixture" curve has large slope at its ENDS.

EQUATIONS OF "MIXTURE" CURVES

Curves of Type in FIGURE 1:

$$\hat{F}(x) = Q_1 F_1(x) + Q_2 F_2(x) \qquad \text{(General Equation)}$$

$$F_1(x) = 1 - \mathrm{EXP}(-(x/\theta_1)^{b_1}) \quad ; \quad F_2(x) = 1 - \mathrm{EXP}(-(x/\theta_2)^{b_2})$$

$$\hat{F}(x) = 1 - Q_1 \mathrm{EXP}(-(x/\theta_1)^{b_1}) - Q_2 \mathrm{EXP}(-(x/\theta_2)^{b_2})$$

$$(Q_1 + Q_2 = 1)$$

Q_1 = Fraction of Mixture from Part (1)

Q_2 = Fraction of Mixture from Part (2) = $1 - Q_1$

A mixture curve of the type in FIGURE 1 (from two intersecting lines) always passes through the INTERSECTION point of the two lines. Furthermore, the formula for the abscissa of the intersection point is

$$x_o = \left(\frac{\theta_2^{\,b_2}}{\theta_1^{\,b_1}} \right)^{\frac{1}{b_2 - b_1}} \qquad (b_2 > b_1)$$

Furthermore, at the intersection point

$$\ln(1 - \hat{F}) = -\left(\frac{\theta_2}{\theta_1} \right)^{\frac{b_1 b_2}{b_2 - b_1}} \qquad (b_2 > b_1)$$

$\hat{F}$ = Ordinate of Mixture Curve

The Weibull slope at any point (of abscissa x) on the mixture curve is

$$\hat{b} = \frac{b_1 Q_1 (1 - F_1) \ln(1 - F_1) + b_2 Q_2 (1 - F_2) \ln(1 - F_2)}{(1 - \hat{F}) \ln(1 - \hat{F})}$$

F_1 = Ordinate of Part (1) at the abscissa x

F_2 = Ordinate of Part (2) at the abscissa x

$\hat{F}$ = Ordinate of Mixture Curve at the abscissa x

$= Q_1 F_1 + Q_2 F_2$

$(Q_1 + Q_2 = 1)$

The Weibull slope of the mixture curve at the intersection point is

$$\hat{b} = b_1 Q_1 + b_2 Q_2 .$$

At the points of the mixture curve having a maximum vertical separation from line (1) of FIGURE 1, the following relation holds:

$$1 - \hat{F} = (1 - F_1)^{\frac{1}{b_1 Q_2 (1 - F_2)}} (1 - F_2)^{\frac{b_2}{b_1^2 Q_1 (1 - F_1)}} .$$

At the intersection point on the mixture curve, the second derivative (on Weibull paper) is

$$Q_1 Q_2 (b_2 - b_1)^2 \left[\left(\frac{\theta_2}{\theta_1} \right)^{\frac{b_1 b_2}{b_2 - b_1}} + 1 \right]$$

$$(b_2 > b_1)$$

In general, the second derivative (on Weibull paper) at any point on the mixture curve is

$$\frac{b_1^2 Q_1 P_1 (1 + \ln P_1) + b_2^2 Q_2 P_2 (1 + \ln P_2) - \left(\frac{1 + \ln P}{P \ln P} \right) \left(b_1 Q_1 P_1 \ln P_1 + b_2 Q_2 P_2 \ln P_2 \right)^2}{\hat{P} \ln \hat{P}}$$

where $P_1 = 1 - F_1(x)$; $P_2 = 1 - F_2(x)$

$$\hat{P} = 1 - \hat{F}(x) = Q_1 P_1 + Q_2 P_2$$