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PLOTTING FAILURES PER MACHINE IN SERVICE FOR SYSTEMS OBEYING THE ULTIMATE VALUE LAW

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INTRODUCTION

Whenever we are concerned with a complex system, which can fail in a multitude of different modes, it turns out that the distribution function for times to failure is not a simple straight line on Weibull probability paper. As a matter of fact, the Weibull plot starts out with a low slope at the early stages and ends up with a steep slope at the upper stages. This is true even for debugged systems. From experience with complex systems it has been found that such systems have a MAXIMUM LIFE (i.e., an ULTIMATE VALUE for hours of endurance) . It is this existence of a maximum life (ultimate value) which causes the Weibull plot to steepen at higher percentiles of the life distribution. For this reason it makes more sense to use the ULTIMATE VALUE MATHEMATICAL FUNCTION in analyzing certain complex systems. The purpose of this bulletin is to describe a method of plotting failures per machine from field data by employing the ULTIMATE VALUE MATHEMATICAL THEORY. In this way we can fit straight lines to data on failures per machine for machines which obey the ULTIMATE VALUE LAW. Such systems would not yield straight lines for LOG-LOG Plots of failures per machine.

BASIC MATHEMATICAL MODEL

The ULTIMATE VALUE CUMULATIVE DISTRIBUTION FUNCTION for first failures of a system has the following form:

$$F(X) = EXP \left[1 - \left(\frac{U}{X} \right)^{\gamma} \right] \quad (1)$$

where

X = Hours of Service (or other time units)

U = Ultimate Value (Maximum Life)

γ = Shape Parameter (Slope of Plot on Ultimate Value Paper)

$F(X)$ = Fraction of Machines in the Population which have failed at least once in X hours.

In order to come up with a mathematical formula for FAILURES PER MACHINE in X hours, we must derive a formula for ENTROPY from equation (1).

ENTROPY at time X is defined as the value of $\ln \frac{1}{1 - F(X)}$.

If we denote ENTROPY by the symbol $\mathcal{E}$, then

$$\mathcal{E} = \ln \frac{1}{1 - F(X)}$$

or $$F = 1 - EXP(-\mathcal{E})$$

Thus, (1) can be written as

$$1 - \mathrm{EXP}(-\varepsilon) = \mathrm{EXP}\left[1 - \left(\frac{U}{X}\right)^{\gamma}\right] \qquad (2)$$

Taking the logarithm of both sides of (2):

$$\ln(1 - \mathrm{EXP}(-\varepsilon)) = 1 - \left(\frac{U}{X}\right)^{\gamma}$$

Transposing:

$$1 - \ln(1 - \mathrm{EXP}(-\varepsilon)) = \left(\frac{U}{X}\right)^{\gamma}$$

Inverting:

$$\frac{1}{1 - \ln(1 - \mathrm{EXP}(-\varepsilon))} = \left(\frac{X}{U}\right)^{\gamma} \qquad (3)$$

Taking the logarithm of both sides of (3):

$$\ln\left[\frac{1}{1 - \ln(1 - \mathrm{EXP}(-\varepsilon))}\right] = \gamma \ln X - \gamma \ln U \qquad (4)$$

Equation (4) is a LINEAR RELATION between $\ln X$ and $\ln\left[\frac{1}{1 - \ln(1 - \mathrm{EXP}(-\varepsilon))}\right]$,

where

X = Time in Service

ε = Failures Per Machine

CONSTRUCTION OF SYSTEMS PLOTTING PAPER

From equation (4) we can construct a plotting grid which will give straight lines for the data on failures per machine in a complex machine (system) obeying the ultimate value law. This is done by making the abscissa scale a LOG scale (for LN X) , and by making the prdinate scale $\ln\left[\frac{1}{1 - \ln(1 - EXP(-\mathcal{E}))}\right]$, where the ENTROPY $\mathcal{E}$ is equivalent to the failures per machine in service time X. The logarithmic abscissa scale presents no difficulty. However, the ORDINATE SCALE must be calibrated in accordance with the following table:

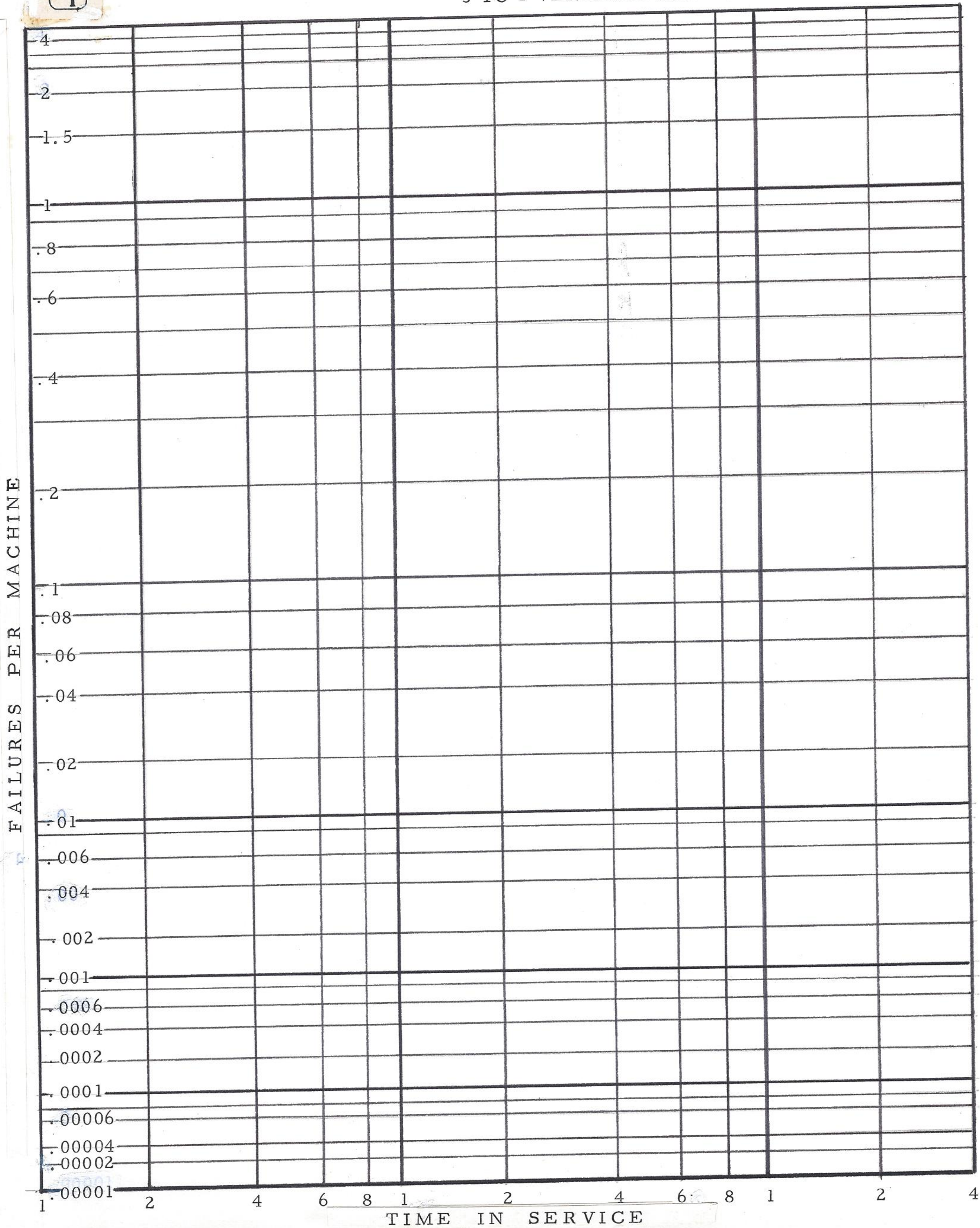
FAILURES PER MACHINE $\mathcal{E}$	VALUE OF THE TRANSFORMED ORDINATE $\ln\left[\frac{1}{1 - \ln(1 - EXP(-\mathcal{E}))}\right]$
.00001	- 2.52676
.00002	- 2.469775
.00003	- 2.434862
.00004	- 2.409343
.00005	- 2.389085
.00006	- 2.372223
.00007	- 2.357741
.00008	- 2.345024
.00009	- 2.333671

FAILURES PER MACHINE $\mathcal{E}$	VALUE OF THE TRANSFORMED ORDINATE $\ln\left[\frac{1}{1 - \ln(1 - EXP(-\mathcal{E}))}\right]$	FAILURES PER MACHINE $\mathcal{E}$	VALUE OF THE TRANSFORMED ORDINATE $\ln\left[\frac{1}{1 - \ln(1 - EXP(-\mathcal{E}))}\right]$
.0001	- 2.323406	.1	- 1.209607
.0002	- 2.253110	.2	- .996126
.0003	- 2.209579	.3	- .854511
.0004	- 2.177503	.4	- .746514
.0005	- 2.151896	.5	- .658945
.0006	- 2.130477	.6	- .585490
.0007	- 2.112003	.7	- .522561
.0008	- 2.095721	.8	- .467887
.0009	- 2.081136	.9	- .419917
.001	- 2.067907	1.0	- .377529
.002	- 1.976246	1.5	- .225128
.003	- 1.918486	2.0	- .135766
.004	- 1.875405	2.5	- .082179
.005	- 1.840679	3.0	- .049808
.006	- 1.811398	3.5	- .030202
.007	- 1.785967	4.0	- .018317
.008	- 1.763413	4.5	- .011109
.009	- 1.743099	5.0	- .006738
.01	- 1.724580	5.5	- .004087
.02	- 1.593716	6.0	- .002479
.03	- 1.508848	6.5	- .001503
.04	- 1.444282	7.0	- .000912
.05	- 1.391438	7.5	- .000553
.06	- 1.346321	8.0	- .000335
.07	- 1.306725	8.5	- .000203
.08	- 1.271294	9.0	- .000123
.09	- 1.239130	9.5	- .000075

By constructing a horizontal LOG SCALE and a vertical scale in accordance with the preceding table of TRANSFORMED ORDINATES yields us the grid shown on page 7 .

SYSTEM ENTROPY GRID

ULTIMATE VALUE MODEL
3 TO 1 VERTICAL EXPANSION



A NUMERICAL EXAMPLE

We are given a set of failure data on a certain collection of machines out in the field. These machines have seen different hours of service (ages 110 hrs. to 760 hrs.) . The number of failures for machines of these different ages are listed in column (3), while the ages are listed in column (1) , and the number of machines of each age appear in column (2) of the data table below:

DATA TABLE

(1) AGE IN HOURS	(2) NUMBER OF MACHINES	(3) NUMBER OF FAILURES
110	50	5
160	45	10
220	42	15
285	40	20
355	38	26
440	35	32
530	25	30
640	20	32
760	16	37

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ANALYSIS AND GRAPHICAL PLOTTING

From the DATA TABLE in the preceding section we divide each number in column (3) by the corresponding number in column (2) to obtain the FAILURES PER MACHINE corresponding to each age in column (1), as follows:

FAILURES PER MACHINE TABLE

(1) AGE IN HOURS	(3) (2) FAILURES PER MACHINE
110	.100
160	.222
220	.357
285	.500
355	.684
440	.914
530	1.200
640	1.600
760	2.312

We now plot AGE IN HOURS as ABSCISSA and FAILURES PER MACHINE as ORDINATE on our SYSTEM ENTROPY GRID to obtain FIGURE 1, which has a SLOPE PARAMETER of .56* and an ULTIMATE VALUE of 913 hours.

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* NOTE: The SLOPE in FIGURE 1 is measured to be 1.68, i.e., THREE TIMES the TRUE SLOPE PARAMETER .56 .

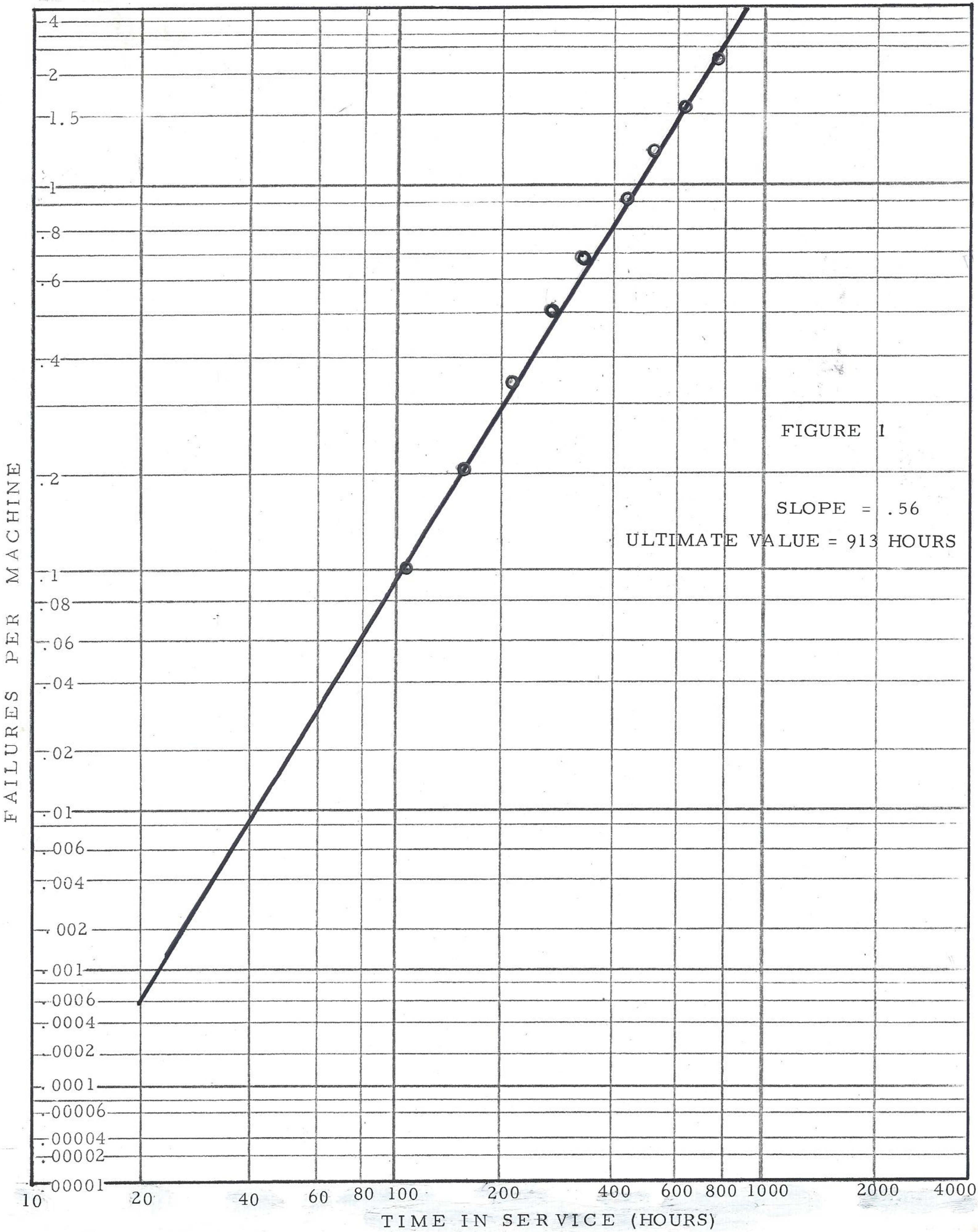
FIGURE 1
SLOPE = .56
ULTIMATE VALUE = 913 HOURS
FAILURES PER MACHINE
TIME IN SERVICE (HOURS)
4
2
1.5
1
.8
.6
.4
.2
.1
.08
.06
.04
.02
.01
.006
.004
.002
.001
.0006
.0004
.0002
.0001
.00006
.00004
.00002
.00001
10
20
40
60
80
100
200
400
600
800
1000
2000
4000