Statistical Bulletin
Reliability & Variation Research

DETROIT RESEARCH INSTITUTE
21900 GREENFIELD ROAD • OAK PARK, MICHIGAN 48237 • (313) 968-1818

LEONARD G. JOHNSON
EDITOR

WANG H. YEE
DIRECTOR

VOLUME 8 BULLETIN 4 AUGUST, 1978

BAYESIAN SUCCESS RUN THEOREMS DERIVED FROM PRIOR DISTRIBUTIONS WITH 50% MINIMUM RELIABILITY

Vol. 8 Bulletin 4 August, 1978

The commonly used conservative success run theorem, which states that the reliability which can be guaranteed with confidence C when N successes are observed in N trials is equal to $R_C(X_0) = (1 - C)^{\frac{1}{N+1}}$, is derived by assuming a prior distribution of reliability which is rectangularly distributed between 0 and 1, as shown in FIGURE 1.

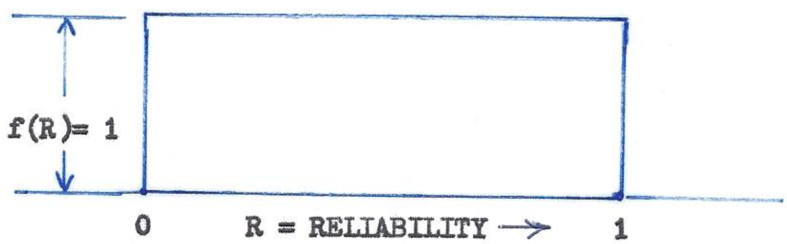

FIGURE 1

Obviously, for most consumer products which have been tested and released for production, this is far too conservative as a prior distribution.

In this bulletin we shall derive some more reasonable success run theorems based on prior distributions which are between .5 and 1.00 on the reliability axis.

One such distribution is the PARABOLIC DISTRIBUTION shown in FIGURE 2.

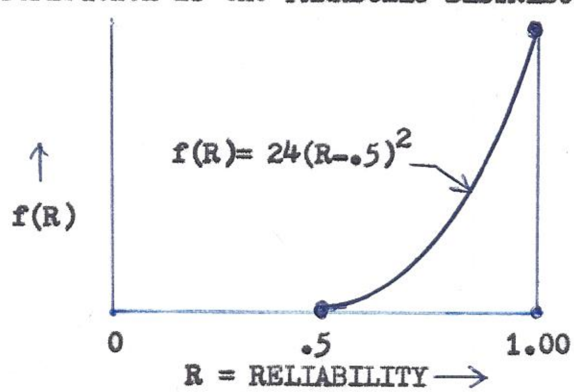

FIGURE 2

Vol. 8 Bulletin 4 August , 1978

SUCCESS RUN THEOREM FOR A PARABOLIC PRIOR DISTRIBUTION

Consider the parabolic prior distribution shown in FIGURE 2. If we carry out the four steps of Bayesian analysis we obtain the following results:

STEP 1: $\text{PROB.}(R \leq \text{Rel.} \leq R + dR) = 24(R - .5)^2\, dR$

STEP 2: $\text{PROB.}\left[(N|N) \quad \text{if } (R \leq \text{Rel.} \leq R+dR) \right] = R^N$

STEP 3: $\text{PROB.}\left[(N|N) \quad \text{and } (R \leq \text{Rel.} \leq R+dR) \right] = 24\, R^N (R - .5)^2\, dR$

STEP 4: $$\text{PROB.}\left[(R \leq \text{Rel.} \leq R + dR) \quad \text{if } (N|N) \right] = \frac{24\, R^N (R - .5)^2 dR}{\int_{.5}^{1} 24\, R^N (R - .5)^2 dR} = g(R)\, dR$$

The confidence C of having a reliability of at least R because of a success run of size N is then

$$C = \int_R^1 g(R)\, dR = \frac{(N^2 + N + 2) - R^N \left[(2R-1)^2 N^2 + (6R-5)(2R-1)N + 2(4R^2 - 6R + 3) \right]}{(N^2 + N + 2) - .5^N} \qquad (I)$$

Let us now evaluate formula (I) for reliability levels of .90, .95, and .99 for various success run sizes from N = 1 to 300. We then obtain the following tables:

SUCCESS RUN TABLES FOR A PARABOLIC PRIOR

For R = .90		For R = .95		For R = .99	
Success Run N	Confidence C	Success Run N	Confidence C	Success Run N	Confidence C
1	.532	1	.302	1	.067
2	.573	2	.333	2	.075
3	.612	3	.363	3	.084
4	.648	4	.393	4	.092
5	.680	5	.421	5	.100
6	.711	6	.448	6	.109
7	.738	7	.475	7	.118
8	.763	8	.500	8	.126
9	.786	9	.524	9	.135
10	.807	10	.547	10	.143
11	.826	11	.569	11	.151
12	.843	12	.591	12	.160
13	.858	13	.611	13	.168
14	.872	14	.630	14	.176
15	.885	15	.648	15	.184
16	.896	16	.665	16	.192
17	.906	17	.682	17	.200
18	.916	18	.698	18	.208
19	.924	19	.713	19	.216
20	.932	20	.727	20	.224
30	.976	30	.836	30	.298
40	.992	40	.902	40	.365
50	.997	50	.941	50	.425
		100	.995	100	.652
				200	.873
				300	.953

SUCCESS RUN THEOREM FOR A QUARTIC PRIOR DISTRIBUTION

Let us next assume a prior distribution between .5 and 1.00 on the reliability axis which is of degree 4 (quartic), as shown in FIGURE 3.

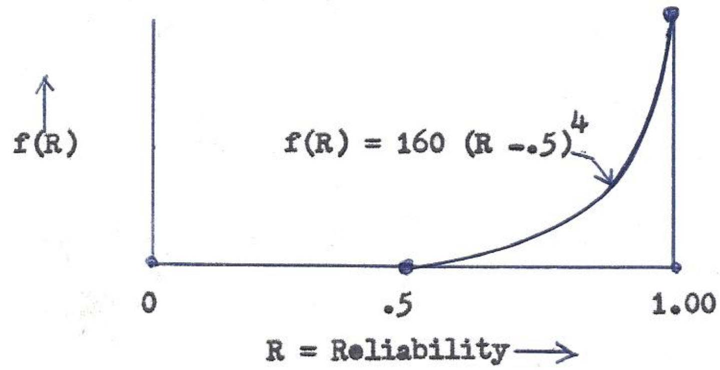

FIGURE 3

Carrying out the four steps of Bayesian analysis for a success run of size N, we obtain the following results:

STEP 1: PROB.$(R \leq \text{Rel.} \leq R + dR) = 160\,(R - .5)^4\,dR$

STEP 2: PROB. $\left[(N|N) \text{ if } (R \leq \text{Rel.} \leq R + dR)\right] = R^N$

STEP 3: PROB. $\left[(N|N) \text{ and } (R \leq \text{Rel.} \leq R + dR)\right] = 160\,R^N (R - .5)^4 dR$

STEP 4: PROB. $\left[(R \leq \text{Rel.} \leq R + dR) \text{ if } (N|N)\right] = \dfrac{160 R^N (R-.5)^4 dR}{\int_{.5}^{1} 160\, R^N (R-.5)^4 dR} = g(R)\,dR$

The confidence C of having a reliability of at least R because of a success run of size N is then

$$C = \int_R^1 g(R)\,dR = \frac{\left(\frac{1}{N+5} - \frac{2}{N+4} + \frac{1.5}{N+3} - \frac{.5}{N+2} + \frac{.0625}{N+1}\right) - R^{N+1}\left(\frac{R^4}{N+5} - \frac{2R^3}{N+4} + \frac{1.5R^2}{N+3} - \frac{.5R}{N+2} + \frac{.0625}{N+1}\right)}{\left(\frac{1}{N+5} - \frac{2}{N+4} + \frac{1.5}{N+3} - \frac{.5}{N+2} + \frac{.0625}{N+1}\right) - .5^{N+5}\left(\frac{1}{N+5} - \frac{4}{N+4} + \frac{6}{N+3} - \frac{4}{N+2} + \frac{1}{N+1}\right)} \quad (II)$$

When we evaluate formula (II) for reliability levels of .90, .95, and .99 for various success run sizes N = 1 to 300 we obtain the following tables:

SUCCESS RUN TABLES FOR A QUARTIC PRIOR

FOR R = .90

Success Run N	Confidence C
1	.707
2	.741
3	.762
4	.782
5	.801
6	.819
7	.836
8	.852
9	.866
10	.879
11	.891
12	.901
13	.911
14	.920
15	.928
16	.935
17	.941
18	.947
19	.952
20	.957
30	.985
40	.995
50	.998

FOR R = .95

Success Run N	Confidence C
1	.438
2	.469
3	.492
4	.514
5	.536
6	.558
7	.579
8	.599
9	.618
10	.637
11	.654
12	.671
13	.687
14	.703
15	.717
16	.731
17	.744
18	.757
19	.769
20	.780
30	.868
40	.921
50	.952
100	.996

FOR R = .99

Success Run N	Confidence C
1	.104
2	.114
3	.122
4	.130
5	.138
6	.146
7	.154
8	.162
9	.170
10	.178
11	.186
12	.194
13	.202
14	.210
15	.218
16	.225
17	.233
18	.241
19	.248
20	.255
30	.326
40	.390
50	.448
100	.666
200	.878
300	.955

DRI STATISTICAL BULLETIN
Vol. 8 Bulletin 4 August , 1978

COMMENTS

By examining the success run tables which we have evaluated in this bulletin we see that

(A) FOR A PARABOLIC PRIOR: To guarantee 90% reliability with 90% confidence we need a success run N = 16. This is 5 units less than the success run N = 21 required by the conventional rectangular prior.

(B) FOR A QUARTIC PRIOR: To guarantee 90% reliability with 90% confidence we need a success run N = 12. This is 9 units less than the success run N = 21 required by the conventional rectangular prior.