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MODAL SPLIT THEORY IN TRANSPORTATION

ELECTRIC CIRCUIT ANALOG

The theory of the splitting of electric current amongst wires in parallel is very simple and depends only upon the resistances of the several wires which current is flowing from point A to point B through three wires whose resistances are R_1 ohms, R_2 ohms, and R_3 ohms, respectively.

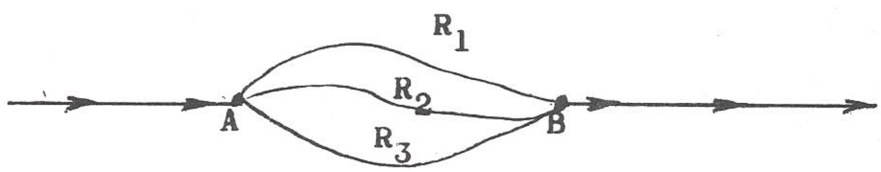


FIGURE 1

Let I_1 = amperes of current in wire #1 (with resistance R_1)
Let I_2 = amperes of current in wire #2 (with resistance R_2)
Let I_3 = amperes of current in wire #3 (with resistance R_3)

According to the basic law of electricity (Ohm's Law) we know that the "IR drop" between A and B is the same along each wire. Hence,

$$I_1R_1 = I_2R_2 = I_3R_3 \quad (1)$$

The total current from A to B is $(I_1 + I_2 + I_3)$ amperes.

Therefore, the three resistances R_1, R_2, and R_3 in parallel can be looked upon as being equivalent to some resultant resistance $\hat{R}$, where

$$I_1R_1 = I_2R_2 = I_3R_3 = (I_1 + I_2 + I_3)\hat{R} \tag{2}$$

From (2) we see that

$$\frac{I_1}{I_1 + I_2 + I_3} = \frac{\hat{R}}{R_1} \tag{3}$$

$$\frac{I_2}{I_1 + I_2 + I_3} = \frac{\hat{R}}{R_2} \tag{4}$$

$$\frac{I_3}{I_1 + I_2 + I_3} = \frac{\hat{R}}{R_3} \tag{5}$$

The ratio (3) represents the fraction of total current flowing through wire #1.

The ratio (4) represents the fraction of total current flowing through wire #2.

The ratio (5) represents the fraction of total current flowing through wire #3.

By adding (3), (4), and (5) we obtain the relation

$$1 = \frac{\hat{R}}{R_1} + \frac{\hat{R}}{R_2} + \frac{\hat{R}}{R_3} \tag{6}$$

Solving (6) for R we obtain the formula

$$\hat{R} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \tag{7}$$

Formula (7) is the standard one for combining three resistances in parallel.

APPLYING THE ELECTRIC ANALOG TO TRANSPORTATION MODES

The simple configuration of three wires in parallel is analogous to <u>three modes of transportation</u> between two geographical points , A and B , in the following fashion:

1. The three currents (I_1, I_2, I_3) are analogous to three volumes (V_1, V_2, V_3) , i.e., numbers of passengers, carried by three modes of transportation between point A and point B.

2. The three resistances (R_1, R_2, R_3) are analogous to three total dollar costs per passenger, which can be expressed as follows:

   For Mode #1: $(\text{Total Cost})_1 = F_1 + T_1 \nu_1 = R_1$

   where, F_1 = passenger fare on Mode #1

   T_1 = Total travel time on Mode #1 (hours)

   ν_1 = Dollar value a passenger on Mode #1 puts on each hour of elapsed travel time

   For Mode #2: $(\text{Total Cost})_2 = F_2 + T_2 \nu_2 = R_2$

   where, F_2 = passenger fare on Mode #2

   T_2 = Total travel time on Mode #2 (hours)

   ν_2 = Dollar value a passenger on Mode #2 puts on each hour of elapsed travel time

For Mode #3: $(\text{Total Cost})_3 = F_3 + T_3 \nu_3 = R_3$

where, F_3 = passenger fare on Mode #3

T_3 = Total travel time on Mode #3 (hours)

ν_3 = Dollar value a passenger on Mode #3 puts on each hour of elapsed travel time

Using equations (3), (4), and (5), we readily come up with a solution to the modal split problem, as follows:

$$\frac{V_1}{V_1 + V_2 + V_3} = \frac{\hat{R}}{R_1} = \frac{\dfrac{1}{(\text{Total Cost})_1}}{\dfrac{1}{\text{Total Cost}_1} + \dfrac{1}{\text{Total Cost}_2} + \dfrac{1}{\text{Total Cost}_3}}$$

This last expression represents the fraction of all passengers which will use Mode #1. It can be written as follows in terms of the F'_is, T'_is, and ν'_is:

$$\frac{V_1}{V_1 + V_2 + V_3} = \frac{\dfrac{1}{F_1 + T_1\nu_1}}{\dfrac{1}{F_1 + T_1\nu_1} + \dfrac{1}{F_2 + T_2\nu_2} + \dfrac{1}{F_3 + T_3\nu_3}} \tag{8}$$

In a similar fashion , we can write

$$\frac{V_2}{V_1 + V_2 + V_3} = \frac{\dfrac{1}{F_2 + T_2\nu_2}}{\dfrac{1}{F_1 + T_1\nu_1} + \dfrac{1}{F_2 + T_2\nu_2} + \dfrac{1}{F_3 + T_3\nu_3}} \qquad (9)$$

$$\frac{V_3}{V_1 + V_2 + V_3} = \frac{\dfrac{1}{F_3 + T_3\nu_3}}{\dfrac{1}{F_1 + T_1\nu_1} + \dfrac{1}{F_2 + T_2\nu_2} + \dfrac{1}{F_3 + T_3\nu_3}} \qquad (10)$$

Equations (8), (9), and (10) represent a complete solution to the modal-split problem in terms of fares (F_1, F_2, F_3) and total elapsed travel times (T_1, T_2, T_3) for the three modes, and the dollar values (ν_1, ν_2, ν_3) which a typical passenger on each mode assigns to each hour of elapsed time on that mode.

From (8), (9), and (10) we see that the volumes on the three modes are distributed according to the proportion

$$V_1 : V_2 : V_3 = \left(\frac{1}{F_1 + T_1\nu_1}\right) : \left(\frac{1}{F_2 + T_2\nu_2}\right) : \left(\frac{1}{F_3 + T_3\nu_3}\right) \qquad (11)$$

Therefore ,

$$V_1(F_1 + T_1 \mathcal{V}_1) = V_2(F_2 + T_2 \mathcal{V}_2) = V_3(F_3 + T_3 \mathcal{V}_3) \qquad (12)$$

Equations (12) simply state that the split-up amongst modes is of such a nature that the cost totals are the same for all modes, when we total up the costs for all passengers on each mode.

The volume ratio between any two modes (say, Mode 2 and Mode 1) is inversely proportional to the TOTAL COST PER PASSENGER. Thus ,

$$\frac{V_2}{V_1} = \frac{F_1 + T_1 \mathcal{V}_1}{F_2 + T_2 \mathcal{V}_2} \qquad (13)$$

MODAL-SPLIT RATIO FOR ANY PAIR OF MODES

From the parallel circuit analog it can be seen that the MODAL-SPLIT RATIO between any pair of modes (which can be denoted by subscripts 1 and 2 without any loss of generality) is given by (13), regardless of the number of modes in parallel between points A and B.

Let ρ = Modal-split ratio between Mode 1 and Mode 2

Then ,

$$\rho = \frac{V_2}{V_1} = \frac{F_1 + T_1 \mathcal{V}_1}{F_2 + T_2 \mathcal{V}_2} = \frac{\text{Total Cost}_1}{\text{Total Cost}_2} \qquad (13)'$$

DETERMINING THE VARIANCE OF THE MODAL-SPLIT RATIO

The fares F_1 and F_2 can be considered fixed when we attempt to determine the variance of the Modal-Split Ratio ρ .

Let σ_ρ^2 = Variance of ρ

Let $\sigma_{T_1}^2$ = Variance of Total Travel Time T_1

Let $\sigma_{\mathcal{V}_1}^2$ = Variance of a passenger's dollar value for each hour of elapsed travel time on Mode 1

Let $\sigma_{T_2}^2$ = Variance of Total Travel Time T_2

Let $\sigma_{\mathcal{V}_2}^2$ = Variance of a passenger's dollar value for each hour of elapsed travel time on Mode 2

Let $\sigma_{\text{total cost}_1}^2$ = Variance of Total Cost_1

Let $\sigma_{\text{total cost}_2}^2$ = Variance of Total Cost_2

Since ρ is a function of F_1, F_2, T_1, T_2, $\mathcal{V}_1$, $\mathcal{V}_2$, we can write

$$\rho = \rho(F_1, F_2, T_1, T_2, \mathcal{V}_1, \mathcal{V}_2) \tag{14}$$

If we consider the fares F_1 and F_2 as constants, we can determine the variance of ρ from the theory of error propagation, as follows:

$$\sigma_\rho^2 = \left(\frac{\partial \rho}{\partial T_1}\right)^2 \sigma_{T_1}^2 + \left(\frac{\partial \rho}{\partial T_2}\right)^2 \sigma_{T_2}^2 + \left(\frac{\partial \rho}{\partial \mathcal{V}_1}\right)^2 \sigma_{\mathcal{V}_1}^2 + \left(\frac{\partial \rho}{\partial \mathcal{V}_2}\right)^2 \sigma_{\mathcal{V}_2}^2 \tag{15}$$

Taking $\rho = \dfrac{F_1 + T_1 \nu_1}{F_2 + T_2 \nu_2}$ we obtain, by differentiation,

$$\frac{\partial \rho}{\partial T_1} = \frac{\nu_1}{F_2 + T_2 \nu_2} \quad ; \quad \frac{\partial \rho}{\partial T_2} = \frac{-\nu_2 (F_1 + T_1 \nu_1)}{(F_2 + T_2 \nu_2)^2}$$

$$\frac{\partial \rho}{\partial \nu_1} = \frac{T_1}{F_2 + T_2 \nu_2} \quad ; \quad \frac{\partial \rho}{\partial \nu_2} = \frac{-T_2 (F_1 + T_1 \nu_1)}{(F_2 + T_2 \nu_2)^2}$$

Substituting these partial derivatives into (15), we obtain

$$\sigma_\rho^2 = \frac{(\nu_1^2 \sigma_{T_1}^2 + T_1^2 \sigma_{\nu_1}^2)(F_2 + T_2 \nu_2)^2 + (\nu_2^2 \sigma_{T_2}^2 + T_2^2 \sigma_{\nu_2}^2)(F_1 + T_1 \nu_1)^2}{(F_2 + T_2 \nu_2)^4} \quad (16)$$

From (16) we can also derive the following expressions for the variance of ρ :

$$\sigma_\rho^2 = \frac{(\nu_1^2 \sigma_{T_1}^2 + T_1^2 \sigma_{\nu_1}^2) + \rho^2 (\nu_2^2 \sigma_{T_2}^2 + T_2^2 \sigma_{\nu_2}^2)}{(F_2 + T_2 \nu_2)^2} \quad (17)$$

or,

$$\sigma_\rho^2 = \frac{\sigma^2_{\text{TOTAL COST}_1} + \rho^2 \sigma^2_{\text{TOTAL COST}_2}}{(\text{TOTAL COST}_2)^2} \quad (18)$$

Thus, knowing the variances of travel times T_1 and T_2 , as well as the variances of dollar values per hour v_1 and v_2, we can determine the variance of the Modal-Split ratio ρ between any two modes (denoted by subscripts 1 and 2).

The relationship (13)' between the MODAL-SPLIT RATIO $\rho = \frac{V_2}{V_1}$.

and the TOTAL COST RATIO $\gamma = \frac{F_2 + T_2 v_2}{F_1 + T_1 v_1}$ is shown graphically in Figure 2 on logarithmic scales.

Taking the square root of both sides of (17) we obtain the following formula for the <u>standard deviation</u> of the Modal-Split Ratio ρ :

$$\sigma_\rho = \frac{\sqrt{(v_1^2 \sigma_{T_1}^2 + T_1^2 \sigma_{v_1}^2) + \rho^2 (v_2^2 \sigma_{T_2}^2 + T_2^2 \sigma_{v_2}^2)}}{F_2 + T_2 v_2} \qquad (19)$$

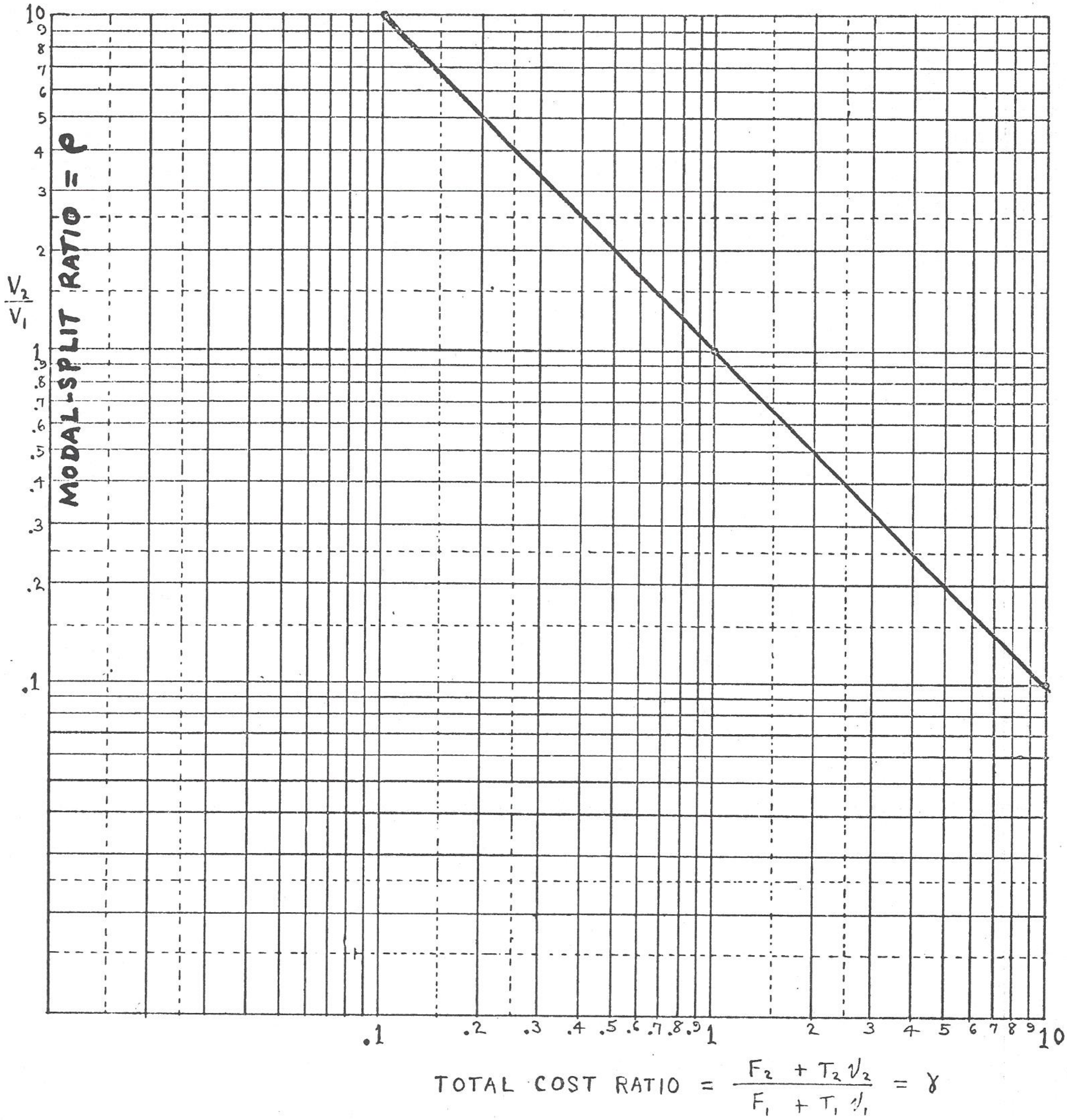


CHART SHOWING THE RELATION BETWEEN MODAL-SPLIT RATIO AND TOTAL COST RATIO

FIGURE 2

NUMERICAL EXAMPLE

In order to illustrate the process of estimating the value of σ_ρ, let us take ν_1 and ν_2 as stochastic quantities from a normal distribution with a mean of 25 cents per hour, and with 3-sigma limits extending from 0 cents per hour to 50 cents per hour, as shown in Figure 3.

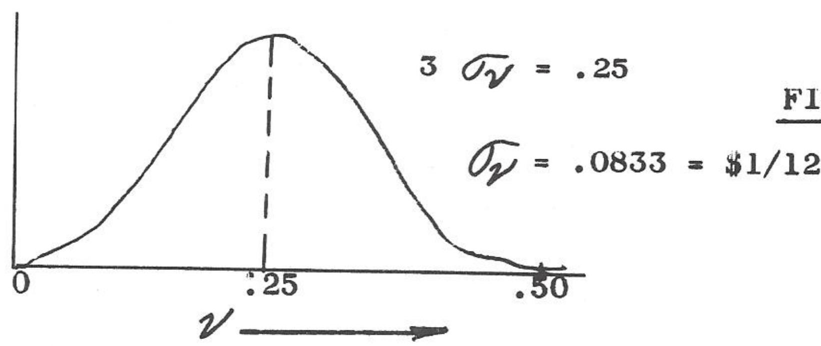
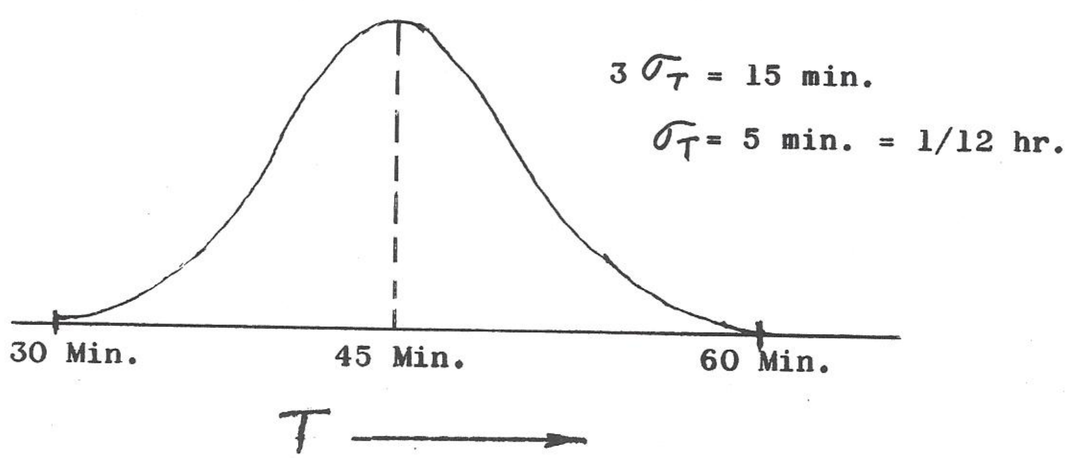


FIGURE 3

Likewise, let us take the travel times T_1 and T_2 as stochastic quantities from a normal distribution whose mean is 45 minutes and whose 3-sigma limits range from 30 minutes to 60 minutes, as shown in Figure 4.

FIGURE 4



Now, in (19), put $\nu_1 = \nu_2 = \$1/4$,

and put $T_1 = T_2 = 3/4$ hr.

Furthermore, put $\sigma_{\nu_1} = \sigma_{\nu_2} = \$1/12$,

and put $\sigma_{T_1} = \sigma_{T_2} = 1/12$ hr.

Then (19) becomes

$$\sigma_\rho = \frac{.06588\sqrt{1+\rho^2}}{F_2 + .1875} \tag{20}$$

The 3-sigma limits on the Modal-Split Ratio ρ then become , using (20),

$$L \leqq \rho \leqq U$$

where

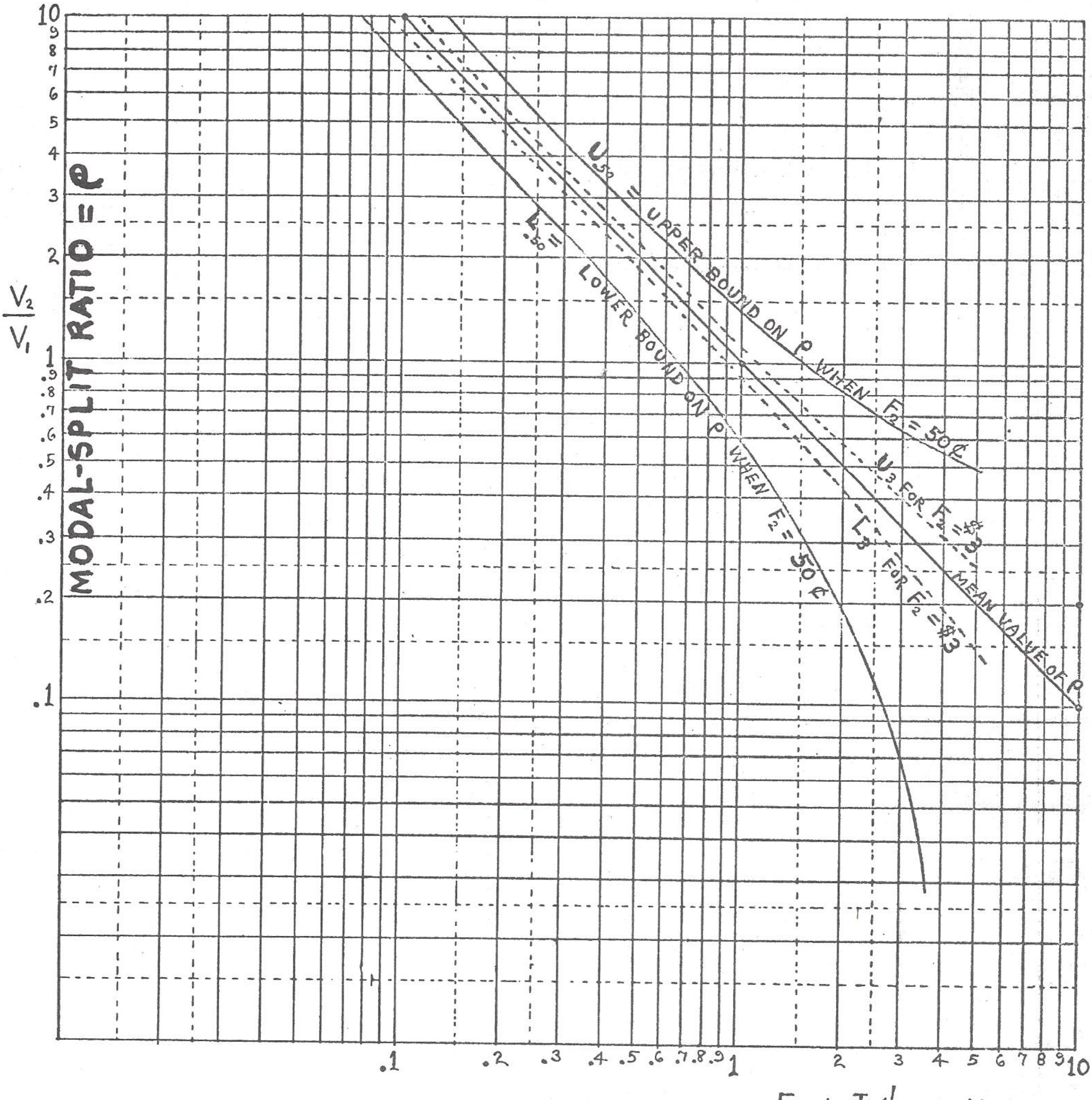
$$L = \rho - 3\left(\frac{.06588\sqrt{1+\rho^2}}{F_2 + .1875}\right) \tag{21}$$

and

$$U = \rho + 3\left(\frac{.06588\sqrt{1+\rho^2}}{F_2 + .1875}\right) \tag{22}$$

The results obtained by evaluating (21) and (22) for different values of ρ when the fare F_2 is taken to be, in one case, 50 cents, and, in another case, \$3, are shown graphically in Figure 5 as the 3-sigma limits on the straight line we constructed in Figure 2.

FIGURE 5



TOTAL COST RATIO $= \dfrac{F_2 + T_2 \nu_2}{F_1 + T_1 \nu_1} = \gamma$

UPPER AND LOWER BOUNDS FOR THE MODAL-SPLIT RATIO vs. THE TOTAL COST RATIO

NOTE: T_1 and T_2 come from a normal population whose 3-sigma limits run from 1/2 hr. to 1 hr.
ν_1 and ν_2 come from a normal population whose 3-sigma limits run from 0 to $1/2 per hr.