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THE ULTIMATE VALUE DISTRIBUTION AND ITS APPLICATIONS

INTRODUCTION

In a good many real life situations we must deal with statistical variables which have a finite upper limit, i.e., a finite ULTIMATE VALUE. In the study of such variables it would be inappropriate to use a distribution function which has a right hand tail extending to infinity. In some cases, an infinite right hand tail cannot even approximate the true situation. Take, for example, the case of scholastic scores, with a maximum possible score of 100%. If the student group which is being scored consists of a lot of smart individuals, it will turn out that most of the scores are equal to or close to 100%. In other words, the MODE of the distribution is near the ULTIMATE VALUE, and then the distribution is abruptly truncated at 100% (the ULTIMATE VALUE).

In other situations, such as in static strength studies of materials, it turns out that there is a finite ULTIMATE STRENGTH to be expected from a test specimen when loaded.

In securities markets there is always an ULTIMATE PRICE LEVEL, which is reached in a bullish phase, from which all subsequent price decline as a bearish phase is entered.

Because of these real life situations it has become necessary to find a suitable mathematical function to fit the facts in such situations. Such a CUMULATIVE DISTRIBUTION FUNCTION is the following :

(Assuming the variable X lies between 0 and the positive Ultimate Value U)

$$F(X) = EXP\left[1 - \left(\frac{U}{X}\right)^{\gamma}\right]$$

where γ is positive and is known as the SHAPE PARAMETER.

A further generalization of F(X) is the so-called 3-parameter ULTIMATE VALUE DISTRIBUTION :

$$F(X) = EXP\left[1 - \left(\frac{U - \alpha}{X - \alpha}\right)^{\gamma}\right]$$

where the variable X is always in the interval ($\alpha \leq X \leq U$).

α = Minimum Value of X

U = Ultimate Value of X

γ = Shape Parameter

PROPERTIES OF THE CDF $F(X) = EXP\left[1 - \left(\frac{U}{X}\right)^{\gamma}\right]$

PROPERTY I : (Median)

$$\text{Median} = \frac{U}{(1 + \ln 2)^{1/\gamma}}$$

PROPERTY II : (B_Q Level)

$$B_Q = \frac{U}{(1 - \ln Q)^{1/\gamma}}$$

PROPERTY III : (Mode)

$$\text{Mode} = \frac{U}{\left(1 + \frac{1}{\gamma}\right)^{1/\gamma}}$$

PROPERTY IV : (Mean and Standard Deviation vs. Shape Parameter)

γ Shape Parameter	$\overline{X}$ Mean	σ Std. Dev.
.2	.1932 U	.2481 U
.25	.2340 U	.2610 U
.33333	.2982 U	.2692 U
.5	.4036 U	.2666 U
1.0	.5963 U	.2190 U
2.0	.7578 U	.1481 U
3.0	.8279 U	.1102 U
4.0	.8661 U	.0875 U
5.0	.8890 U	.0725 U
∞	U	0

RECURRING RELATIONS

For K = An Interger $\geq$ 2 : $\overline{X}_{1/K} = \frac{U}{K-1} - \frac{1}{K-1} \overline{X}_{1/K-1}$

For any $\gamma > 0$: $\sigma_\gamma = \sqrt{U \overline{X}_{\gamma/2} - \overline{X}^2_\gamma}$

MISCELLANEOUS VALUES

$\overline{X}_{1/6}$ = .1614 U

$\overline{X}_{1/7}$ = .1398 U

$\overline{X}_{1/8}$ = .1229 U

$\overline{X}_{1/9}$ = .1096 U

$\overline{X}_{1/10}$ = .0989 U

LINEARIZED PLOTTING OF THE ULTIMATE VALUE CUMULATIVE DISTRIBUTION FUNCTION

The CDF $$F(X) = EXP\left[1 - \left(\frac{U}{X}\right)^{\gamma}\right]$$

can be linearized by taking

$$\xi = \ln X \quad \text{(new abscissa)}$$

and

$$\eta = \ln\left(\frac{1}{1 + \ln\frac{1}{F}}\right) \quad \text{(new ordinate)}$$

Then

$$\eta = \gamma\,\xi + C$$

where $C = -\gamma \ln U$

This linearizing transformation is the mathematical basis for Ultimate Value Probability Paper. (See Figure 1.)

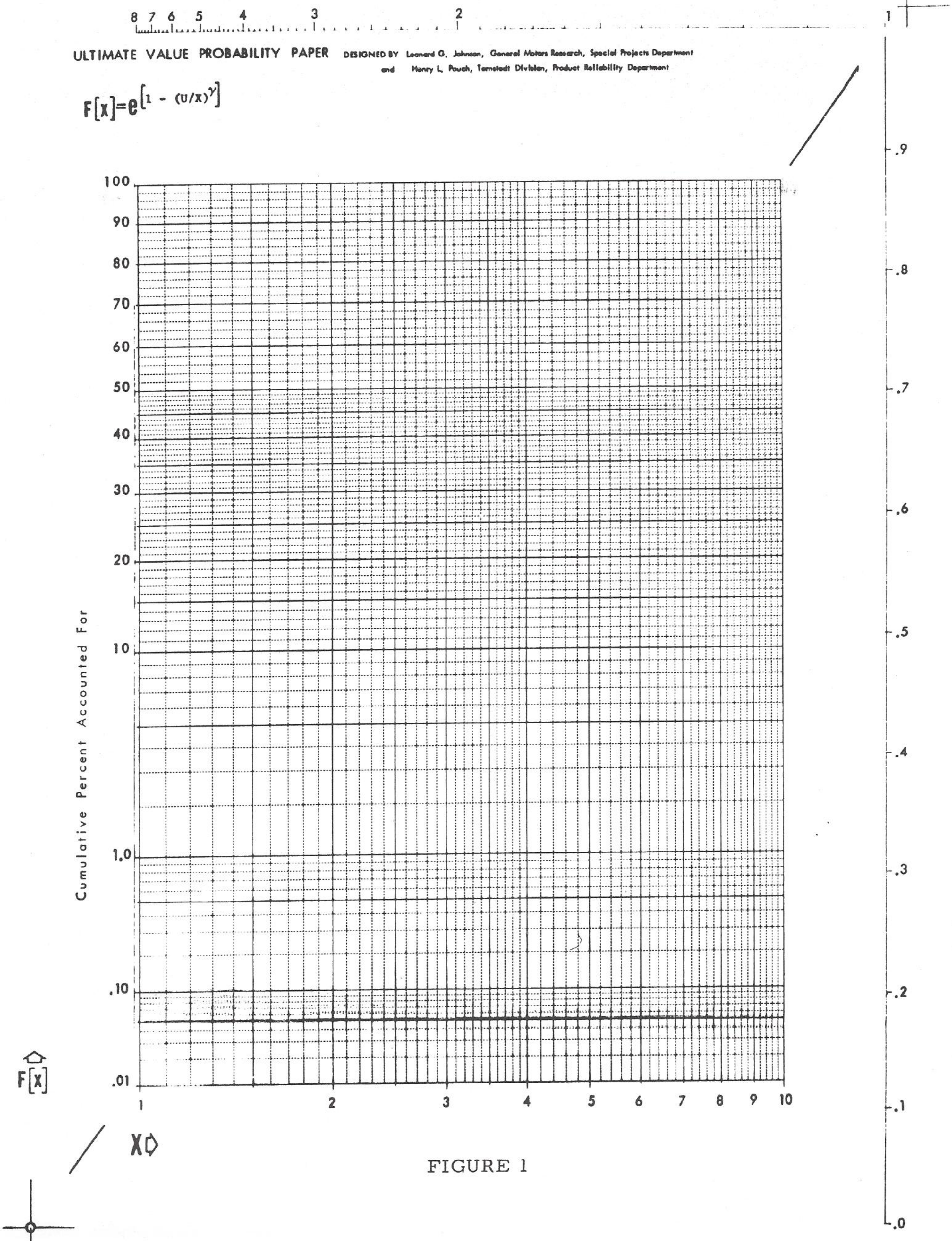
8 7 6 5 4 3 2 1
ULTIMATE VALUE PROBABILITY PAPER
DESIGNED BY Leonard G. Johnson, General Motors Research, Special Projects Department
and Henry L. Pouch, Ternstedt Division, Product Reliability Department
F[X]=e[1 - (U/X)γ]
100
90
80
70
60
50
40
30
20
10
1.0
.10
.01
Cumulative Percent Accounted For
1 2 3 4 5 6 7 8 9 10
F[X]
X
.9
.8
.7
.6
.5
.4
.3
.2
.1
.0

FIGURE 1

ILLUSTRATIVE APPLICATIONS

EXAMPLE #1

Fifteen specimens were loaded until they fractured. The results are shown below :

Specimen #	Lbs. to Fracture
1	630
2	770
3	880
4	940
5	1060
6	1130
7	1220
8	1300
9	1380
10	1470
11	1550
12	1640
13	1720
14	1815
15	1910

To fit an ULTIMATE VALUE DISTRIBUTION to these data we assign MEDIAN RANKS to the fracture loads , as follows:

Lbs. to Fracture (ABSCISSA)	Median Rank (ORDINATE) %
630	4.5%
770	11.0%
880	17.5%
940	24.0%
1060	30. 5%
1130	37.0%
1220	43.5%
1300	50.0%
1380	56.5%
1470	63.0%
1550	69.5%
1640	76.0%
1720	82.5%
1815	89.0%
1910	95.5%

Plotting the abscissas and ordinates on ULTIMATE VALUE PROBABILITY PAPER yields FIGURE 2.

From FIGURE 2 we find that

$$U = \text{ULTIMATE VALUE} = 1995 \text{ lbs.}$$

$$\gamma = \text{Shape Parameter} = 1.22$$

Hence, if X = Fracture Strength, then the CDF of Fracture Strength is

$$F(X) = \text{EXP}\left[1 - \left(\frac{1955}{X}\right)^{1.22}\right]$$

From this,

$$\text{MEDIAN} = \frac{U}{(1 + \ln 2)^{1/\gamma}} = 1296 \text{ lbs.}$$

$$B_{10\%} = \frac{U}{(1 - \ln .1)^{1/\gamma}} = 750 \text{ lbs.}$$

$$B_{90\%} = \frac{U}{(1 - \ln .9)^{1/\gamma}} = 1837 \text{ lbs.}$$

Furthermore,

$$\text{MEAN} = .65\,U = 1296 \text{ lbs.}$$

$$\text{Std. Dev.} = .20\,U = 399 \text{ lbs.}$$

EXAMPLE # 1

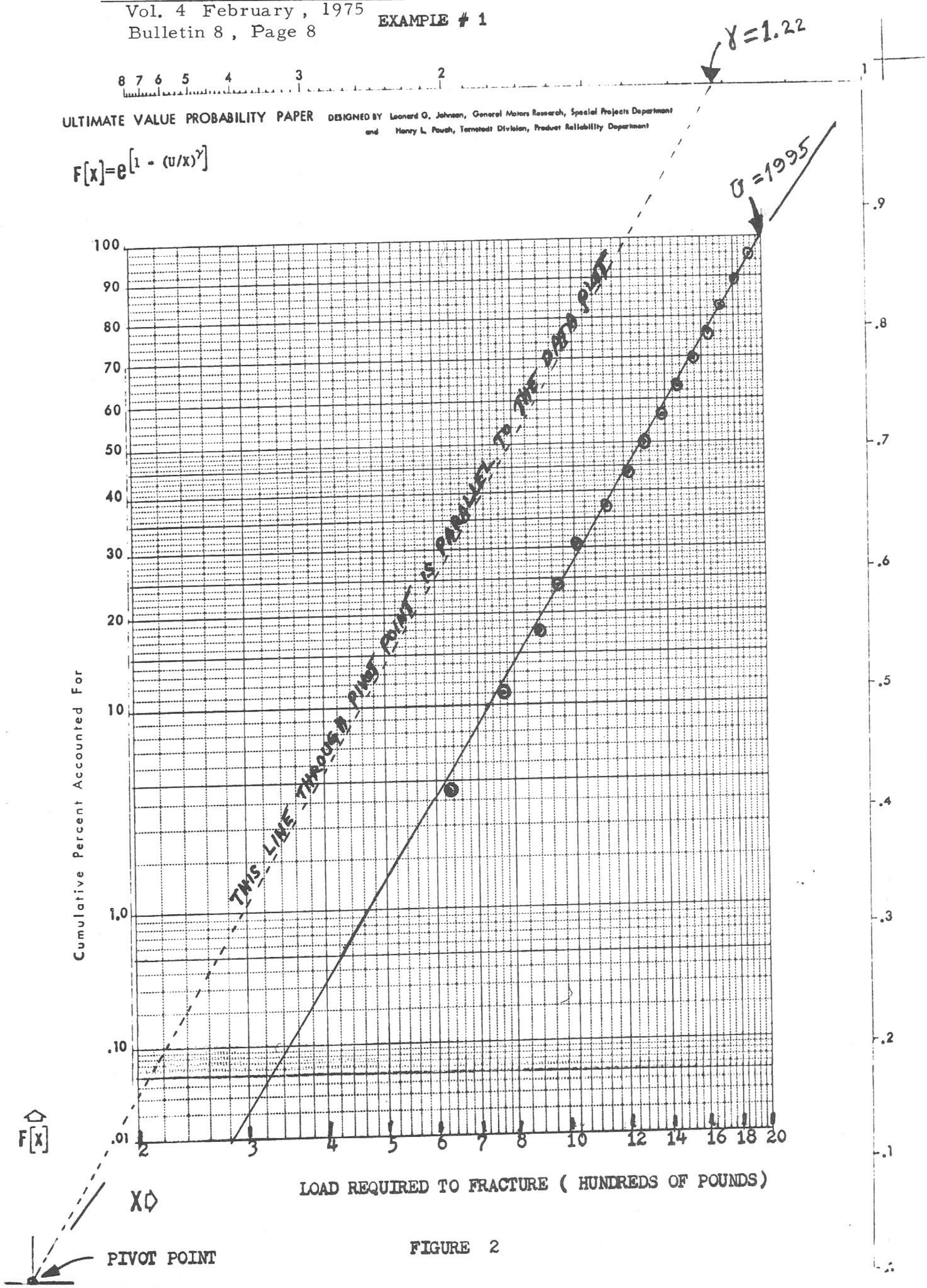

FIGURE 2

EXAMPLE #2

ESTIMATING THE NEXT BULL MARKET GOAL FOR THE DOW JONES INDUSTRIALS

Over the past four years the DJI have shown a shape parameter of 2 (approx.). The LIQUIDITY RATIO between the end of 1974 and the end of 1973 is

$$L = \frac{\text{Vol. Necessary to Change DJI by 1\% (end of '74)}}{\text{Vol. Necessary to Change DJI by 1\% (end of '73)}} = .3$$

(See WALL STREET JOURNAL ----- January 8, 1975)

Taking the SUPPLY CURVE of the DJI as an ULTIMATE VALUE DISTRIBUTION of Shape Parameter 2, we write

$$F(X) = \text{EXP}\left[1 - \left(\frac{U}{X}\right)^2\right]$$

where X = DJI ; U = Ultimate Value (Next Bull Market Goal)

Now, at the end of '73 : X = 851

and, at the end of '74 : X = 616

Hence, from the liquidity ratio expression above :

$$\frac{F(616) - F[616 - .01(616)]}{F(851) - F[851 - .01(851)]} = .3$$

The value of U which satisfies this equation is

$$U = 1209.$$

Hence, we estimate the next BULL MARKET GOAL for the DJI to be 1209 (at least).*

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* We say "at least" because a new bull market would feed on itself, and would thus cause a growth of optimism, which would cause U to increase also.

APPENDIX

THE SHAPE OF THE PROBABILITY DENSITY FUNCTION

From the CDF

$$F(X) = \text{EXP}\left[1 - \left(\frac{U}{X}\right)^{\gamma}\right]$$

we obtain, by differentiation with respect to X, the PDF

$$f(X) = \frac{\gamma}{X}\left(\frac{U}{X}\right)^{\gamma} \text{EXP}\left[1 - \left(\frac{U}{X}\right)^{\gamma}\right]$$

FIGURE 3 shows how the shape of f(X) varies with the value of the Shape Parameter γ.

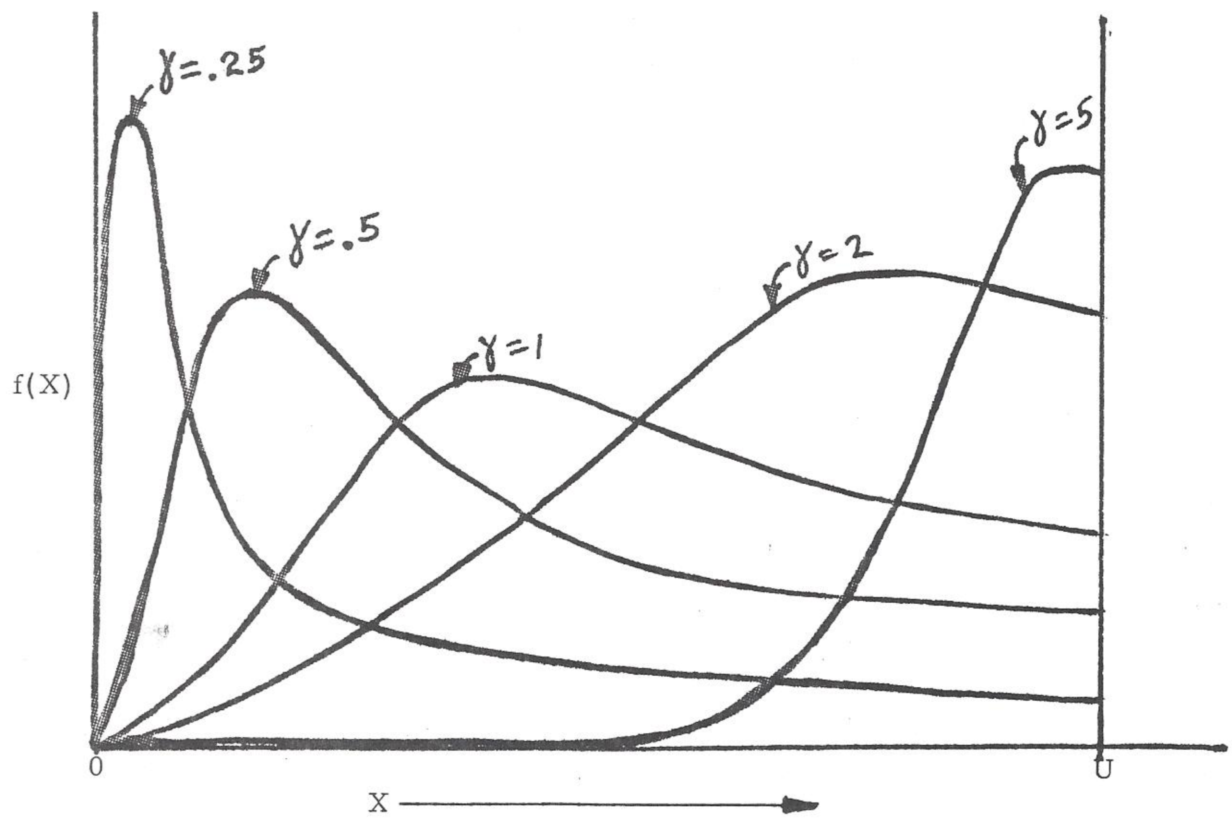

FIGURE 3