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A UNIVERSAL SYSTEM CDF AND ITS NTIC APPROXIMATION

INTRODUCTION

A good many manufacturing firms are in the business of producing systems. Some examples of manufactured systems are :

(1) Automobiles
(2) Agricultural Machines
(3) Computers
(4) Aircraft
(5) Appliances
(6) Medical Instruments
(7) Electric Circuits

In addition to manufactured systems, such as those listed above, there are what are called Natural Systems, such as

(A) The Human Body
(B) Trees and Living Plants
(C) Aquatic Animals
(D) Land Animals

In general, we must distinguish between a system and individual elements or components which make up the system. It is true that what is often considered to be an individual component is so complicated that it is really a system by itself. In such a case we talk of a sub-system which belongs to the TOTAL SYSTEM.

In this discussion we are confining ourselves to total systems and their statistical properties.

STATISTICAL PROPERTIES OF TOTAL SYSTEM LIFE

I : Healthy Systems --- From actual life data on "Healthy" manufactured systems it has been observed that

(a) The Weibull Slope of early failures is unity.
(b) There is a Finite Maximum Life , which cannot be exceeded, even by the best item in the population .

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II: Unhealthy Systems --- A system is defined as UNHEALTHY (or ABNORMAL) if it has a special defect (such as a genetic defect in a new-born child) which causes an abnormally early failures or death (infant mortality) . For such an "unhealthy" system it has been observed that

(a) The Weibull Slope of the earliest failures is less than unity .
(b) A Finite Maximum Life exists .

A UNIVERSAL SYSTEM CDF FOR HEALTHY SYSTEMS

Since a healthy system has an initial Weibull Slope of unity , and possesses a finite maximum life , it follows that the Cumulative Distribution Function (CDF) of the time to system failure must possess mathematical properties consistent with these facts of healthy system life.

If X denotes the time to system failure , and F(X) denotes the cumulative fraction of the population failed in time X , then a possible form for F(X) is the following :

$$F(X) = \frac{e^{-\left(\frac{L-X}{L-\phi}\right)^b} - e^{-\left(\frac{L}{L-\phi}\right)^b}}{1 - e^{-\left(\frac{L}{L-\phi}\right)^b}}$$

where

L = Maximum Life ($L > 0$)
ϕ = Auxiliary Parameter ($\phi < L$)
b = Shape Parameter ($b > 0$)

If this function F(X) is plotted on Weibull Paper, we find that

(a) The initial Weibull Slope (near X = 0) is unity .
(b) The curve is asymptotic to X = L (Maximum Life) at the upper end .
(c) F (0) = 0 and F(L) = 1 .

Therefore, this function serves as a good mathematical model for the distribution of the life of a healthy system . We shall call it the UNIVERSAL HEALTHY SYSTEM CDF .

OTHER PROPERTIES OF THE UNIVERSAL HEALTHY SYSTEM CDF

(1) $$B_Q \text{ Life} = L - (L - \emptyset)\left[\ln \frac{1}{Q + (1 - Q)\, e^{-\left(\frac{L}{L-\emptyset}\right)^{b}}}\right]^{1/b}$$

(2) B_Q Slope (i.e., Weibull Slope at $X = B_Q$)

$$= b \left\{ \frac{B_Q \left(\frac{L - B_Q}{L - \emptyset}\right)^{b-1} \left[(Q + (1 - Q)\, e^{-\left(\frac{L}{L-\emptyset}\right)^{b}} \right]}{(1 - Q)\left(\ln \frac{1}{1 - Q}\right)(L - \emptyset)\left[1 - e^{-\left(\frac{L}{L-\emptyset}\right)^{b}}\right]} \right\}$$

(3) Minimum Life $= 0$

THE SPECIAL CASE OF $\emptyset = 0$

A good many healthy systems have statistical life distributions which can be handled by taking the auxiliary parameter $\emptyset = 0$. In such a case we can write the following formula for the SHAPE PARAMETER b:

$$b = \frac{\ln \ln \left[\frac{1}{Q + (1 - Q)\, e^{-1}}\right]}{\ln(1 - \rho_Q)} \qquad \text{Shape Parameter Formula}$$

where $$\rho_Q = \left(\frac{B_Q}{L}\right) = \left(\frac{B_Q \text{ Life}}{\text{Maximum Life}}\right)$$

STATISTICAL DETERMINATION OF THE SHAPE PARAMETER b FOR A HEALTHY SYSTEM FROM LIFE TEST DATA ON THE SYSTEM

Suppose a sample of K systems is tested to failure, with the resultant set of K ordered times to failure being $(X_1, X_2, X_3, \ldots, X_K)$.

We then fit a POWER FUNCTION CDF (i.e., an NTIC CDF) by finding a LINEAR LEAST SQUARES curve fit to λ ln (i, K) vs. $\ln X_i$ (i = 1 to K), where

$$\lambda(i, K) = \text{Median Rank of the } i^{th} \text{ order statistic in } K$$

$$= \frac{i - .3}{K + .4} \qquad \text{(Benard's Formula)}$$

From such a least squares fit we find that

$$F(X) = \left(\frac{X}{L}\right)^{\eta}$$

where L = Maximum Life and η = Ntic Exponent

The least squares equation in terms of logarithms is $\ln F = \eta \ln X - \eta \ln L$

η = Slope parameter of the linear least squares fit

$-\eta \ln L$ = Intercept of the linear least squares fit

Thus,

$$\ln L = -\frac{\text{INTERCEPT}}{\text{SLOPE}}$$

OR

$$L = \text{EXP}\left(-\frac{\text{INTERCEPT}}{\text{SLOPE}}\right) = \text{MAXIMUM LIFE}$$

The B_Q Life according to the least squares fit is $B_Q = LQ^{1/\eta}$

$$\therefore \quad \rho_Q = \frac{B_Q}{L} = Q^{1/\eta}$$

Now, going back to the SHAPE PARAMTER FORMULA on page 3 for the Universal Healthy System CDF (with $\emptyset = 0$), we find that

$$\text{SHAPE PARAMETER } b = \frac{\ln \ln \left[\dfrac{1}{Q + (1 - Q)\, e^{-1}}\right]}{\ln(1 - Q^{1/\eta})}$$

where η = NTIC SLOPE of the life test data.

More specifically, we can select a particular value of Q , say Q = .9 . Then

$$b = \frac{\ln \ln \left[\dfrac{1}{.9 + .1\, e^{-1}} \right]}{\ln (1 - .9^{1/\eta})} = \frac{-2.728793}{\ln(1 - .9^{1/\eta})}$$

NUMERICAL EXAMPLE OF SYSTEM LIFE DATA

A certain manufactured machine was tested by taking a sample of 10 machines and running them to failure . The results were as follows (in order of life) :

MACHINE #	HRS. TO FAILURE	MEDIAN RANK
1	1101 hrs.	.0673
2	2402	.1635
3	3599	.2593
4	4807	.3558
5	5710	.4519
6	6503	.5481
7	7300	.6442
8	8006	.7404
9	8699	.8365
10	9405	.9327

Plotting these data on LOG-LOG PAPER (For an N-Tic fit) we obtain FIGURE 1 , which shows an NTIC Slope of 1.22 and a MAXIMUM LIFE of L = 10,5000 hours.

Hence , the shape parameter for a UNIVERSAL HEALTHY SYSTEM CDF (with $\emptyset$ = 0) is

$$b = \frac{-2.728793}{\ln (1 - .9^{1/1.22})} = 1.09 \qquad \text{ANSWER}$$

Thus , the SYSTEM CDF is

$$F(X) = \frac{e^{-\left(\frac{10,500 - X}{10,500} \right)^{1.09}} - e^{-1}}{1 - e^{-1}}$$

For reference, FIGURE 2 shows the shapes of such system CDF curves for b = 1, 2, 4, 8, and 16 . The MAXIMUM LIFE is taken as L = 10,000 in all these curves .

LOG-LOG PLOT OF MACHINE FAILURE DATA

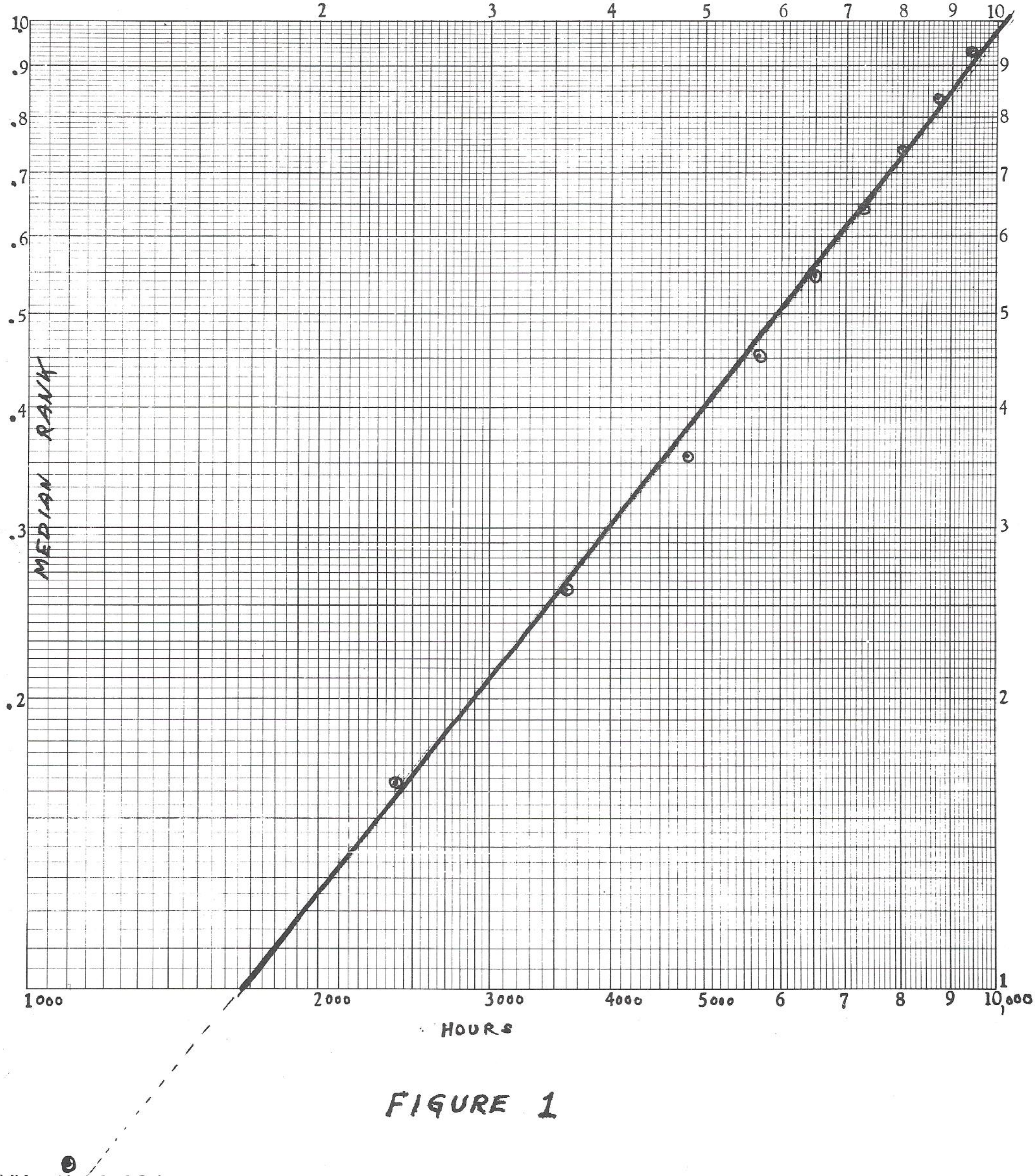

FIGURE 1

Full Logarithmic 1 x 1 Cycles

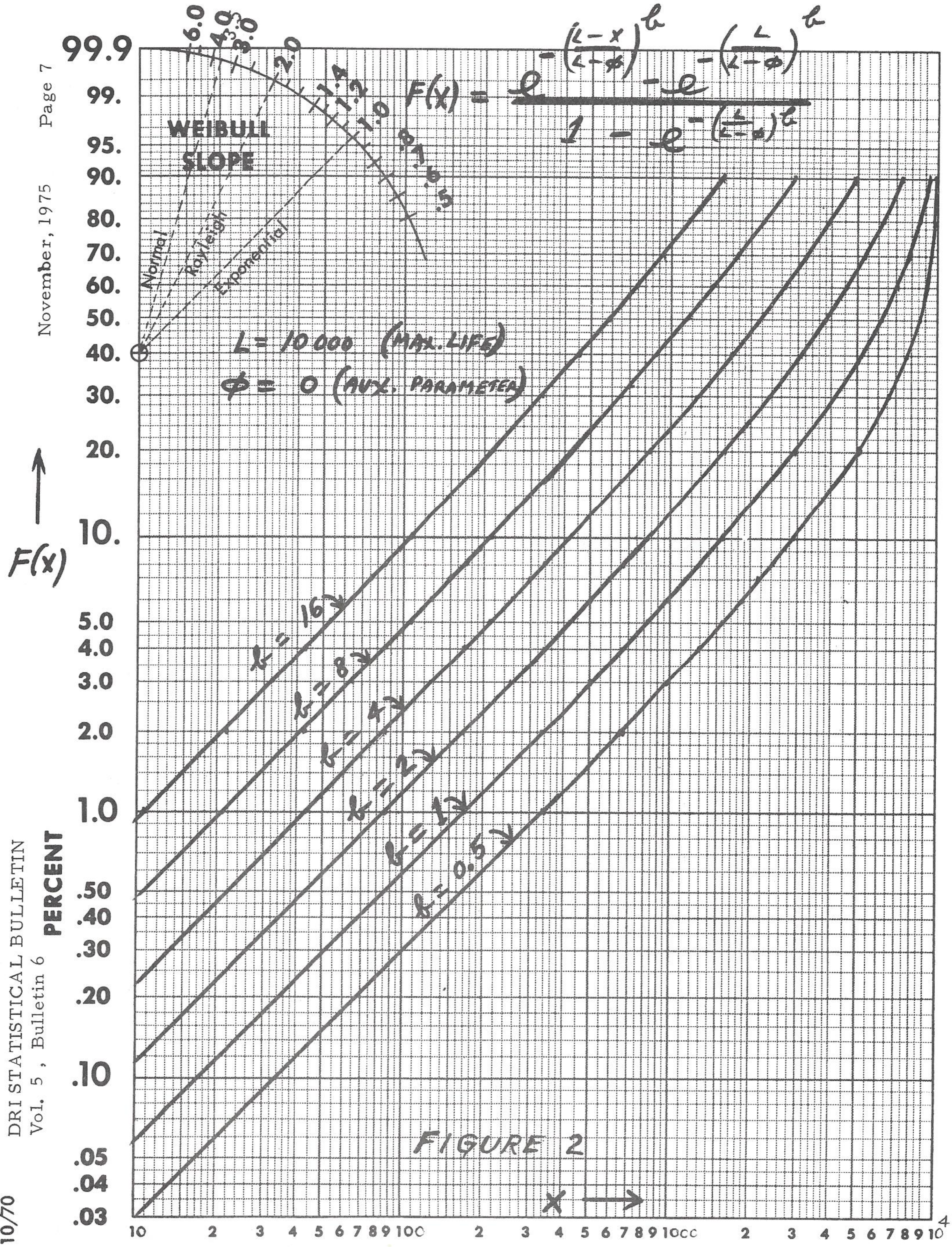

FIGURE 2

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