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THE DRAWBACKS OF EMPLOYING MAXIMUM LIKELIHOOD ESTIMATION IN WEIBULL ANALYSIS

INTRODUCTION

It is well known that the maximum likelihood method of parameter estimation can lead to extremely biased results in many problems. This is especially true for small samples ---- for example , the classical sample estimate of variance of a Gaussian population is biased , and is corrected by changing the divisor in the mean sum of squares to N-1 in place of the maximum likelihood divisor N.

This problem of bias in the maximum likelihood method is even more serious in the case of Weibull parameter estimation . The purpose of this issue of the Statistical Bulletin is to give quantitative empirical formulas for the amount of bias in the maximum likelihood method of Weibull analysis, by taking the more unelegant MEDIAN RANK LEAST SQUARES METHOD as the method which gives reasonable and practical answers .

I : BIAS IN THE WEIBULL SLOPE WHEN THE MAXIMUM LIKELIHOOD METHOD IS USED

Through 25 years of experience with industrial problems of life-testing and reliability prediction in such fields as bearing design , crank shafts , axle shafts , gear design , tire life , engine life , electronic components , clutches , brake linings , bending specimens , torsion bars , and multitudes of other mechanical parts and their failures , we have found that the MEDIAN RANK LEAST SQUARES METHOD on WEIBULL PROBABILITY PAPER is very useful , and gives reasonable results , whereas , the MAXIMUM LIKELIHOOD METHOD can give dangerously misleading results . In the case of the WEIBULL SLOPE , the maximum likelihood estimate is universally too high .

Based on a sample with N failures (regardless of how many are suspended),

the ratio $\left(\dfrac{\text{LEAST SQUARES WEIBULL SLOPE}}{\text{MAXIMUM LIKELIHOOD WEIBULL SLOPE}} \right)$ can be

represented by the empirical function

$$\phi_N = 1 - \text{EXP}\left[- \left(\frac{N - 1.64}{.92585} \right)^{.300952} \right] \qquad (1)$$

(N = Number of Failures ≥ 2)

From FORMULA (1) we can construct TABLE 1 on the next page .

TABLE 1

N	ϕ_N	% BIAS (Max. Likelihood over Least Squares)
2	.5288	89.1 %
4	.7343	36.2 %
8	.8324	20.1 %
16	.8979	11.4 %
32	.9427	6.07 %
64	.9711	2.98 %
128	.9876	1.26 %
256	.9952	0.48 %
512	.9987	0.13 %

According to this table , a sample of at least 150 failures is needed in order to assure us that the MAXIMUM LIKELIHOOD WEIBULL SLOPE will be within 1 % of the MEDIAN RANK LEAST SQUARES WEIBULL SLOPE. FIGURE 1 at the end of this report is based on TABLE 1 and FORMULA (1) .

II : BIAS IN THE SCALE PARAMETER (CHARACTERISTIC LIFE) AS ESTIMATED BY THE METHOD OF MAXIMUM LIKELIHOOD

It can be shown that for a TRUE POPULATION WEIBULL SLOPE of b , the ratio

$$\left(\frac{\text{LEAST SQUARES CHARACTERISTIC LIFE}}{\text{MAX. LIKELIHOOD CHARACTERISTIC LIFE}} \right) = \rho$$

is represented by the formula

*** $$\rho = \frac{N^{1/b}\,(N - 1)!}{\Gamma(N + 1/b)} , \qquad (2)$$

where N = Number of Failures in the sample

NUMERICAL EXAMPLE OF BIAS IN CHARACTERISTIC LIFE

For a Weibull Slope of 2 , and sample size N = 5 failures , the ratio (2) becomes

$$\rho = \frac{5^{1/2}(4)!}{\Gamma(5.5)} = 1.025$$

Thus , in this case, the characteristic life is about 2 1/2 % lower by the maximum likelihood formula , assuming the Weibull slope 2 is correct.

But , according the FORMULA (1) , the MAXIMUM LIKELIHOOD SLOPE for N = 5 would come out 30 % too high , or to a value of $\hat{b} = 2.6$ (instead of the true value 2). This biased $\hat{b}$ would make FORMULA (2) to yield

$$\rho = \frac{5^{1/2.6}(4)!}{\Gamma(5.384615)} \simeq 1.025$$

, still indicating that the characteristic life by the maximum likelihood method would be about 2 1/2 % too low in this particular situation .

CONCLUSION

The MAXIMUM LIKELIHOOD METHOD of parameter estimation for a two-parameter Weibull population should not be used without bias correction. Furthermore , for the minimum life of a three-parameter Weibull function , there is no sensible maximum likelihood estimator available (since MAXIMUM LIKELIHOOD says that the lowest sample value X_1 = population minimum life) , and we might as well use the MEDIAN RANK LEAST SQUARES METHOD with CORRELATION COEFFICIENTS as indices of GOODNESS OF FIT .

***See Page 2 of DRI COMPUTER ANALYSIS BULLETIN VOL. 1 , BULLETIN 1 , SEPTEMBER , 1973 .

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FIGURE 1